Stochastic Resonance in Noisy Non-Dynamical Systems

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We have analyzed the effects of the addition of external noise to non-dynamical systems displaying intrinsic noise, and established general conditions under which stochastic resonance appears. The criterion we have found may be applied to a wide class of non-dynamical systems, covering situations of different nature. Some particular examples are discussed in detail.

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Stochastic resonance (SR) [1–16] is a phenomenon wherein the addition of noise to a system enhances its response to a periodic input signal. This fact is usually characterized by an increase of the output signal-to-noise ratio (SNR) as the noise level increases. The main fingerprint of this constructive role played by noise is the appearance of a maximum in the SNR at a nonzero noise level, although recently new manifestations have been found such as multiple maxima [17] or a divergent SNR with the noise level [18]. In spite of the fact that at the beginning this phenomenon was restricted to bistable systems, nowadays it is known that there are different situations in which SR appears, as for instance in monostable, excitable, and non-dynamical systems.

In regards to non-dynamical systems, SR has only been found in threshold and threshold-like devices [13,19] and in the situation in which the output of the system consists of a random train of identical pulses, with the probability of a pulse generation exponentially depending on the input signal [20]. Non-dynamical systems, however, encompass more general situations, of which many display intrinsic noise whose effects cannot systematically be neglected.

In this Letter we address precisely the problem of establishing general conditions under which SR appears in noisy non-dynamical systems. We will show that the addition of external noise to the periodic input signal may give rise to the enhancement of the response of the system, therefore implying the presence of SR. In this sense, the addition of external noise helps to overcome the limitations imposed by the unavoidable intrinsic noise.

A non-dynamical system can be characterized by its output as a function of given input parameters which do not depend on the output. This very general requirement is responsible for the fact that non-dynamical systems are frequently found in many different scientific areas, encompassing a wide variety of situations, including, to mention just a few: biological membranes, ionic channels, solid state diodes, quantum dots, and self-assembled molecular nanostructures. Usually, the output is a fluctuating quantity which is characterized by its mean value and variance. The characteristics of this intrinsic noise, to some extent, are determined by the state of the system, which in turns depends on the value of the input. This stochastic behavior is reflected in the class of systems described by

$$I(t) = H(V) + \xi(t) \ , \tag{1}$$

where $I(t)$ and V are the output and the input, respectively. The function $H(V) = \langle I(t) \rangle$ corresponds to the deterministic response and $\xi(t)$ represents the intrinsic noise. Usually the characteristic time scales of $\xi(t)$ are smaller than any other entering the system. Under this circumstance, it can be approximated by Gaussian white noise with zero mean and correlation function $\langle \xi(t)\xi(t+\tau) \rangle = G(V)\delta(\tau)$. Here, the function $G(V) = \langle I(t)^2 \rangle - \langle I(t) \rangle^2$ shows the dependence of the noise on the state of the system. Eq. (1) then describes, in general, the behavior of most noisy non-dynamical systems. For instance, in the case of systems consisting of spike trains with Poisson statistics, the variance is equal to the mean, then $G(V) = H(V)$, and $H(V)$ gives the pulse rate generation.

In order to analyze how the addition of external noise affects the response of the system to a slow periodic signal, we consider the input consisting of sinusoidal and random contributions, namely $V \equiv V(t) = V_s(t) + V_r(t)$. Here $V_s(t) = \alpha \sin(\omega_0 t)$, with α being the amplitude and $\omega_0/2\pi$ the frequency. The random term $V_r(t)$ is assumed to be Gaussian noise with zero mean and correlation function $\langle V_r(t)V_r(t+\tau) \rangle = \sigma^2 \exp(-\tau/\tau_F)$, where σ^2 defines the noise level and τ_F is the correlation time, which is assumed $\tau_F \ll 2\pi/\omega_0$. The SNR can be computed from the averaged power spectrum

$$P(\omega) = 4 \int_0^{2\pi/\omega_0} \frac{\omega_0}{2\pi} dt \int_0^{\infty} \langle I(t)I(t+\tau) \rangle_{\xi,r} \cos(\omega\tau) d\tau \ , \tag{2}$$

which follows from the correlation function

$$\begin{aligned} \langle I(t)I(t+\tau) \rangle_{\xi,r} &= \langle H(V(t))H(V(t+\tau)) \rangle_r \\ &\quad + \langle G(V(t)) \rangle_r \, \delta(\tau) \ . \end{aligned} \tag{3}$$

Here $\langle . \rangle_{\xi,r}$ and $\langle . \rangle_r$ indicate average over both noises and only over $V_r(t)$, respectively. By considering sufficiently small amplitudes of the input signal and low noise level, $H(V)$ and $G(V)$ can be expanded in power series of V, therefore

$$\mathrm{SNR} = \frac{2\pi\alpha^2 H'(H' + H'''\sigma^2)}{2G + (4\tau_F {H'}^2 + G'')\sigma^2} \ , \tag{4}$$

where $'$ indicates the derivative of the function with respect to its argument [21]. All the functions have been evaluated at $V = 0$.

Concerning the value of H', two different situations can be analyzed. If $H' \neq 0$ and the inequality

$$2H'''G - 4\tau_F {H'}^3 - H'G'' > 0 \tag{5}$$

holds, the SNR is an increasing function of the noise level, which reveals that the addition of noise enhances the response of the system to a weak periodic signal. As usually occurs, if the SNR decreases for high noise level, then the SNR presents at least a maximum, thus indicating the appearance of SR.

As a first example illustrating the applicability of our results, we will analyze the particular case discussed in Ref. [20] in our context. In these circumstances $G(V) = H(V) = r(0)\exp(V)$, where $r(0)$ is the pulse rate generation at $V = 0$. Then, Eq. (5) indicates that SR appears for $r(0)\tau_F < 1/4$, in agreement with Ref. [20,22]. Note that for small amplitudes of the input signal, following our approach [Eqs. (1), (2) and (3)], the SNR can analytically be computed giving the same result as the one of Ref. [20,22].

In order to show the great generality of Eq. (5) we will analyze a system in which the internal noise does not follow Poisson statistics, as is the case, for instance, of a model for electrical conduction which displays saturation. In this model, $I(t)$ corresponds to the current intensity and V to an input voltage. To be explicit, we will consider

$$H(V) = \frac{V + V_0}{R[1 + (V + V_0)^2]^{1/2}} \ , \tag{6}$$

$$G(V) = \frac{Q}{[1 + (V + V_0)^2]^{1/2}} \ , \tag{7}$$

where R and Q are constants. Here V_0 represents a constant bias voltage. For low values of the total applied voltage $(V + V_0)$, the system behaves as a linear resistor. For high values, however, the non-linear terms become important to the extent that the current intensity saturates. Semiconductors systems displaying hot electrons effects constitute a well known example of systems exhibiting this non-linear behavior [23]. According to our previous analysis [Eq. (5)], this system exhibits SR for sufficiently large values of V_0. In Fig. (1) we have depicted the SNR as a function of the noise level, for different values of V_0. These results have been obtained by numerically averaging over realizations of the noise terms. SR appears for sufficiently large values of V_0, as predicted by the criterion established in Eq. (5). The occurrence of SR is not exclusive of this particular model. We have analyzed other models exhibiting saturation [24] and found that SR is a common phenomenon.

Let us now discuss the case in which $H' = 0$. In this situation, the SNR vanishes for $\sigma^2 = 0$. Therefore, for low noise level and sufficiently small amplitudes of the input signal, if $H(V)$ is not symmetric around $V = 0$, the SNR must be an increasing function of σ^2, explicitly [25]

$$\mathrm{SNR} = \frac{\pi\alpha^2 {H'''}^2 \sigma^4}{4G} \ . \tag{8}$$

One then concludes that, when $H' = 0$, SR is always present for sufficiently small amplitudes of the input signal and its appearance does not depend on the form of noise $\xi(t)$, provided that the noise term is different from zero at $V = 0$. This situation may occur in many systems of interest, as for instance in tunnel diodes [26,24], ionic channels [28], and in a general way in any system exhibiting maxima, minima or inflection points in the I-V characteristics.

As an example of a system for which $H' = 0$, that can be solved analytically, for any value of the amplitude and noise level, we will consider the case

$$H(V) = V^3 \ , \tag{9}$$

$$G(V) = Q \ , \tag{10}$$

where Q is a constant. The SNR is found to be

$$\mathrm{SNR}=\pi\frac{18\sigma^4\alpha^2+9\sigma^2\alpha^4+\frac{9}{8}\alpha^6}{2Q+\tau_F(44\sigma^6+54\sigma^4\alpha^2+\frac{27}{2}\sigma^2\alpha^4)}\quad . \tag{11}$$

From the previous result we can elucidate how the SNR behaves in all the range of values of σ^2 and α^2 . It is interesting to point out that for sufficiently small amplitudes of the input signal and low noise level ($\alpha \ll \sigma^2 \ll 1$), Eq. (11) reduces to Eq. (8), then the SNR $\sim \sigma^4$ is an increasing function of σ^2 and SR appears, as predicted by our previous general analysis. For high noise level the behavior is SNR $\sim \sigma^{-2}$, then it decreases when increasing the noise level. Another notable feature is that the SNR is an increasing function of the noise level when τ_F is sufficiently small, irrespective of the value of the amplitude of the input signal. In Fig. (2) we have depicted the SNR corresponding to Eq. (11) as a function of the noise level, for different values of τ_F. This figure clearly illustrates the appearance of a maximum in the SNR at nonzero noise level and the dependence of the SNR on the correlation time of the input noise, namely the output of the system is enhanced when decreasing τ_F.

Another interesting situation in which $H'=0$ occurs is when the I-V curve exhibits a plateau, as for instance in the case of a quantum dot displaying Coulomb gap [27]. Under these circumstances, an input of finite amplitude is required before any effect on the output is produced. Added noise helps small deterministic signals to reach this finite amplitude. Therefore, the output may be enhanced giving rise to the appearance of SR.

Finally, in order to elucidate how the addition of noise enhances the output in the class of systems we are considering, we have shown in Fig. (3) two time series corresponding to zero and the optimum noise levels for the explicit case given through Eqs (9) and (10). This figure clearly shows that the addition of an optimal amount of noise makes the presence of the input signal manifest.

Our analysis has established general conditions in which SR emerges in noisy non-dynamical systems. In this way, we have shown that the appearance of SR in non-dynamical systems is not a particular situation occurring in threshold devices and spike-train systems with exponential rate generation, as believed up to now. On the contrary, SR is a robust phenomenon that may occur in many different physical, chemical and biological non-dynamical systems. One aspect we would like to emphasize is that a simple expression we have found is sufficient to make the presence of SR manifest. This general criterion, which refers to a wide class of non-dynamical systems, may directly be applied to new cases, thereby extending the scope and perspectives of SR.

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FIG. 1. SNR for the system described through Eqs. (6) and (7) as a function of σ for different voltages: $V_0 = 0$ (filled circles), $V_0 = 0.5$ (empty circles), $V_0 = 1.0$ (filled squares), $V_0 = 1.5$ (empty squares), and $V_0 = 2$ (filled triangles). The values of the remaining parameters are $Q = 0.1$, $R = 1$, $\tau_F = 0.1$, $\alpha = 0.32$ and $\omega_0/2\pi = 0.1$.

FIG. 2. SNR [Eq. (11)] for the system described through Eqs. (9) and (10) as a function of σ for different correlation times of the input noise. The values of the parameters are $Q = 0.1$, $\alpha = 0.3$, $\tau_F = 1$ (continuous line), $\tau_F = 0.3$ (dotted line), $\tau_F = 0.1$ (dashed line), and $\tau_F = 0.03$ (dot-dashed line).

FIG. 3. (a) Input periodic signal (b) Time evolution of $\tilde{I}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} I(s)ds$ corresponding to Eqs. (9) and (10), with $\alpha = 0.4$, $\omega_0/2\pi = 1$, $Q = 10^{-5}$, $\Delta t = 2.4 \times 10^{-4}$, $\tau_F = 10^{-4}$ and $\sigma^2 = 0$. (c) Same situation as in (b) but $\sigma^2 = 0.4$.